Cremona's forgotten curve

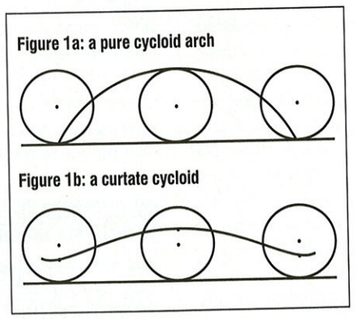
Could a simple system involving a wheel, a pencil and a piece of paper be the basis of the great Cremonese makers' arching designs? **Quentin Playfair** investigates the curtate cycloid curve

I am wary of theories about violins and violin sound. All too often the theoretician ends up maintaining that one particular focus of interest is the only significant feature of an instrument and all other aspects of what is really a complex and interdependent mechanism hold mere supporting roles.

What follows is simply something I have noticed – an observation that may or may not be related to sound, but which is interesting to me because it indicates a common system that was incorporated into the technique of the Cremonese makers without restricting their individual styles.

A cycloid curve derives its name from the Greek word for wheel, *cyclos*. Anyone who played with a spirograph as a child will know how many curves and shapes can be derived from a point within a moving circle. Included in the general category of cycloid are several more specific terms, of which the one I will be discussing is known to the mathematician as a curtate cycloid. In introducing this type of curve the exact formula is less important than the visualisation of what the curve is and how it is made; however, a mathematical formula is provided on page 1197.

Imagine a wheel resting on a straight edge. Behind it is a sheet of paper. Put a pencil through a hole in the centre of the wheel, rotate the wheel along the straight edge and you will draw a line parallel to it. Repeat this with the pencil on a point on the edge of the wheel and you will get a pure cycloid arch (see figure 1a). But if you put the pencil through a hole that is not at the centre or the edge and roll the wheel, something like figure 1b will be traced by the pencil.

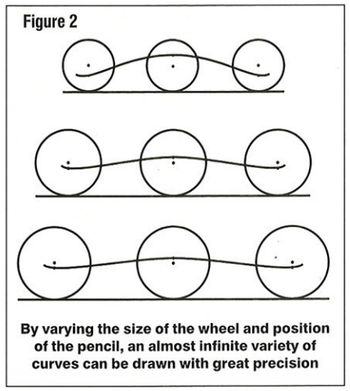


This is a curtate cycloid. A little thought will confirm that the size of the wheel determines the width of the curve and that the distance of the hole from the centre establishes the height of the curve – the further from the centre, the higher the resulting arch. By varying height and width an almost infinite variety of curves can be drawn with great precision (see figure 2), although the tools for constructing a curve of desired dimensions could not be simpler – an appropriately sized wheel and a pencil.

While longitudinal arching does seem to have common features in most golden-period Cremonese instruments, I cannot find a single, common source for the longitudinal arching of either the backs or the generally flattened curve of the tables. The cross-arching area between the *f*-holes seems individual too, following a pattern more akin to a section of a cylinder than anything else. But all other cross-archings of the greatest Cremonese makers, including the Amati and Guarneri families as well as Stradivari, match curtate cycloid curves with remarkable accuracy. The obvious differences between them are the product of two choices that were the decision of the maker: first, the longitudinal arching, and consequently the height at the centre of any given cross-arch; second, the distance between the two lowest points of the arch, often found not at the purfling but at a point well inside it.

Until I started noticing a common thread, I would have been sceptical had anyone proposed that a single system could be used to embrace the majority of the classical makers' arching. Examining two instruments whose dimensions have been given in STRAD posters helps to demonstrate the system (see figure 3). The cross-arching of the upper backs of both the 1649 'Alard' Nicolò Amati (STRAD March 1992) and the 1735 Peter Guarneri of

Diagrams: Quentin Playfair



By varying the size of the wheel and position of the pencil, an almost infinite variety of curves can be drawn with great precision

Venice (STRAD December 1995) rise 9mm when measured from the lowest point of the arch. The total widths of the backs at this point are 162mm and 165mm respectively. These are very similar figures. Yet the appearance of the two violins at this point of arching is very different, and this difference lies in the position of the lowest points of the arching. The Amati scoops down before rising and the actual arch occupies only 134mm of the 162mm width. The Guarneri has almost no scoop. The arching starts right from the edge and takes up 158mm of the 165mm width.

In figure 3 the two arching patterns given in the posters are copied in black. Immediately above them, in red, are computer-generated cycloid curves, sized to match the arching, which are in all essentials identical to those that can be made with a wheel. They are produced from two variable figures – the height at the centre of the arch and the distance between the lowest points of the arch – just as the wheel-created curves can be varied by changing the distance of the hole from the wheel centre and the wheel's circumference. Both makers show individuality in the way they approach a 9mm arch in a space of rather more than 160mm. But, as the red lines indicate, both makers' styles can be matched with curtate cycloid curves. The difference between the two is thus not in the geometry of their arching but in their choice of how to use it.

If correct, the hypothesis proposed here – that the Cremonese makers obtained their variety by individual use of a common geometric construction – it should be possible to recreate the templates they used by drawing curtate cycloids of appropriate width and height and checking them against the originals. Figures 4–7 show archings from many Cremonese instruments of varied sizes. These are printed in black, while their corresponding templates are printed in red and are curtate cycloid curves. Comparison between the actual archings and their geometric counterparts is easy. I have produced half-templates,

Figure 3a: Peter Guarneri of Venice violin, 1735 (illustrated above)

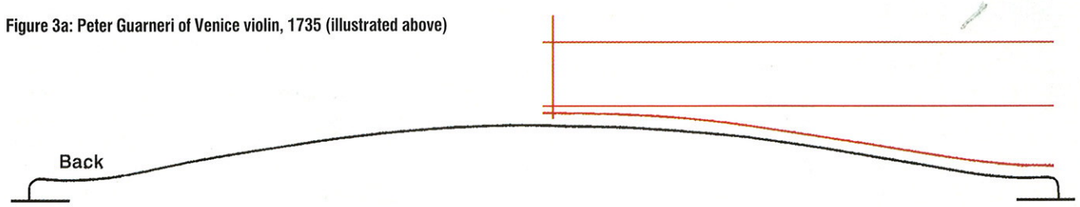


Figure 3b: Nicolò Amati 'Alard' violin, 1649

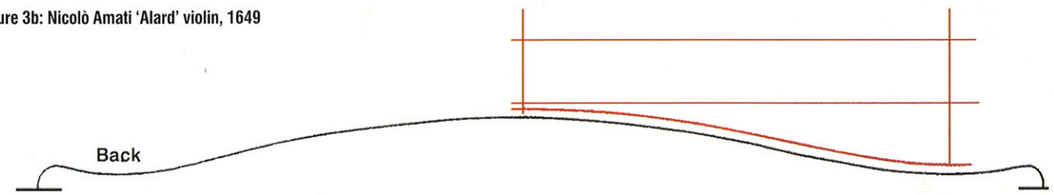


Photo: Richard Valencia

Photo: Clarissa Bruce

following Roger Hargrave's discussion of these in *The Working Methods of Guarneri del Gesù* (*Giuseppe Guarneri del Gesù* vol.II, Peter Biddulph, London 1998). For reasons of space the archings of the larger instruments are also confined to half.

I have not included the arch of the table at the bridge point, which seems to follow rules of its own, and have avoided instruments which are obviously distorted, although it has to be borne in mind that no old instrument is exactly the same shape as it was when it left the workbench.

Given that there does appear to be a link between the makers who represent the backbone of the Cremonese school and the use of curtate cycloids, it seems worthwhile to investigate the historical background of this apparently obscure curve, not only within the mathematical community but also among the general educated public in the 16th and 17th centuries. I have looked at all kinds of cycloid, since it is impossible to study one without being aware of the others.

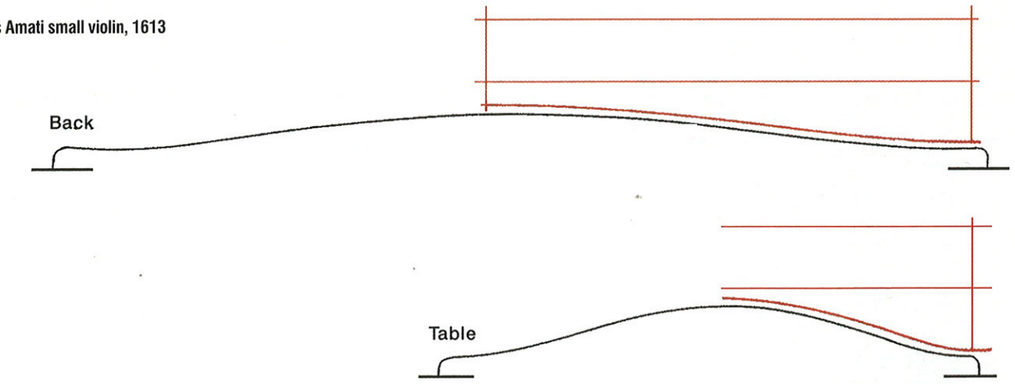
While we know that the ancient Greeks had discovered cycloid curves, they did not have the mathematical knowledge to analyse them. The first significant use of cycloids was by the second-century astronomer Ptolemy. He proposed an earth-centred universe in which the heavenly bodies moved in epicycloids, which are the complex curves produced when a wheel is rotated around another wheel. His theory held for 1,300 years, and consequently the use and formation of cycloids was common knowledge to the educated during that time. However, as late as 1451, when Cardinal Nicholas of Cusa sought to investigate cycloids, he was still unable to establish a formula. The shapes were known, the interest was there, but the mathematical skills were lacking.

This was not a difficulty for the German painter Albrecht Dürer (1471–1528), whose concerns were more pragmatic. He was not a mathematical scholar in the sense of Pascal or Newton but, typically for his time, he believed in the close relationship of art and geometry. Between 1525 and his death Dürer wrote extensively on geometry, but with the practical element in mind. Certainly he was aware of cycloid curves and even ventured into the more rarefied topic of epicycloids. His work was not intended to break intellectual ground; it was a practical and detailed explanation of how geometry could benefit artists and craftsmen.

By the end of the 16th century things had begun to change. Mathematics had started to catch up with the curve that was so easy to make and so hard to analyse. Up to this point those who had written about it had done so descriptively, and it was Galileo (1564–1642) who first used the term cycloid. His suggestion that a pure cycloid would make a suitable arch for a bridge was put into effect many years later in the construction of the Ponte di Mezzo over the Arno in Florence. Others also studied the curves, including Marin Mersenne (1588–1648) and Pascal (1623–62), who wrote his last mathematical work on cycloids and offered a prize for anyone who could devise formulae for certain properties of the shape. Christopher Wren (1632–1723) was among many prominent thinkers to attempt a solution, while Liebniz (1646–1716) and Newton (1643–1727) independently investigated the curious property of a cycloid inverted into a cup-like shape – regardless of where a ball is released on it, it always takes the same amount of time to reach the bottom.

By the end of the 17th century cycloid curves had been analysed and exploited in as many ways as the imagination of the time could devise. Among many examples, the Dutch physicist Christiaan Huygens had used their properties to develop clocks of remarkable accuracy, and a French engineer named Desargues proved their value as the basis for efficient gear systems. So it is not surprising to see the term emerge when, in 1726, Jonathan Swift poked fun at the Royal Society and scientists in general. He strands the luckless hero of *Gulliver's Travels* in a community whose prime concern is always mathematics, even in the way the food is cooked: 'In the first course, there was a shoulder of mutton cut into an equilateral triangle, a piece of beef into a rhomboides, and a pudding into a cycloid.'

Figure 4: Brothers Amati small violin, 1613



It is generally assumed that our technical data and skills are increasing year by year. We find it difficult to imagine that our knowledge could be less than that of our ancestors, so it is hard to believe that a geometric construction that sounds so obscure in the present could have been fairly well known in the past.

While it is hardly likely that the violin makers of Cremona revealed their trade methods to the world in general, we can be sure that, like any intelligent workers, they took in anything useful the world had to offer them. From Dürer in 1525, near the time of Andrea Amati's birth, to Jonathan Swift in 1726, when Stradivari's life was drawing to a close, the cycloid family of curves was a part of every informed person's general knowledge, whether it was as a mathematical formula, an engineering tool or simply an easy and interesting way to create an ornamental shape. It seems to me that asking: How could they have been aware of such a recondite aspect of geometry? is invalid. The appropriate question is: How did we forget such a simple and elegant construction?

There are two ways to generate curtate cycloid curves. The first is the traditional method with a wheel. The second involves a computer programme that calculates and then draws the line a wheel would trace. To make the wheels I used an adjustable hole-cutter and 1/4 inch plywood. The hole-cutter is adjusted so that the circumference of the wheel it produces is the same as the width of the desired curve.

THE MATHEMATICS

If we confine ourselves to the path traced by a point along the radius of a circle rolling along a flat surface, there are three kinds of cycloid curve: the pure cycloid, curtate cycloid and prolate cycloid. These can all be expressed as the locus of a point P, at a distance h from the centre of a circle whose radius is a, and whose centre is O. The varying angle of the line OP in relation to the flat surface on which the wheel moves is represented by t.

Thus: $x = at - h\sin(t)$ $y = a - h\cos(t)$
When h = a the result is a pure cycloid
When h < a the result is a curtate cycloid
When h > a the result is a prolate cycloid (a curve with loops, technically called cusps)

Information on cycloid curves is available on the internet, including the University of St Andrews site: www-groups.dcs.st-and.ac.uk/~history/Curves/Cycloid.html

Figure 5: Stradivari 'Archinto' viola, 1669 (illustrated opposite)